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Reduction of quantum phase fluctuations implies antibunching of photon

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Abstract

A clear physical meaning of the Carruthers–Nieto symmetric quantum phase fluctuation parameter (U) has been provided in Susskind–Glogower and Barnett–Pegg formalism of quantum phase and it is shown that the reduction of phase fluctuation parameter U with respect to its coherent state value corresponds to an antibunched state. Thus nonclassicality of a state may be manifested through the phase fluctuation parameters. As examples, quantum phase fluctuations in different optical processes, such as four wave mixing, six wave mixing and second harmonic generation have been studied by using Carruthers–Nieto quantum phase fluctuation parameters. The operators required for the calculation of quantum phase fluctuations are expressed in closed analytical forms (up to second order in coupling constant). It is also found that the reduction of phase fluctuations compared to their initial values are possible in all the three cases studied here. Thus nonclassical (antibunched) state exists in all these cases.

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1. Introduction

The phase is almost omnipresent in physics. But the introduction of Hermitian phase operators have some ambiguities (interested readers can see the reviews [1–3]) which lead to many different formalisms [4–6] of quantum phase problem. Among these different formalisms, Susskind–Glogower (SG) [4] and Barnett–Pegg (BP) [5] formalisms played most important role in the studies of phase properties and the phase fluctuations of various physical systems. For example, SG formalism has been used by Fan and Zaidi [7], Sanders et al. [8], Yao [9], Gerry [10], Carruthers and Nieto [11] and many others to study the phase properties and phase fluctuations. On the other hand, Lynch [12,13], Vaccaro and Pegg [14], Tsui [15], Pathak and Mandal [16] and others have used the BP formalism for the same purpose.

Commonly, standard deviation of an observable is considered to be the most natural measure of quantum fluctuation [17] associated with that observable and the reduction of quantum fluctuation below the coherent state level corresponds to a nonclassical state. For example, an electromagnetic field is said to be electrically squeezed field if uncertainties in the quadrature phase observable X reduces below the coherent state level (i.e. $(\Delta X)^2 < \frac{1}{2}$) and antibunching is defined as a phenomenon in which the fluctuations in photon number reduces below the Poisson level (i.e. $(\Delta N)^2 < \langle N\rangle$) [18,19]. Standard deviations can also be combined to form some complex measures of nonclassicality, which may increase with the increasing nonclassicality. As an example, we can note that the total noise of a quantum state which, is a measure of the total fluctuations of the amplitude, increases with the increasing nonclassicality in the system [17]. Particular parameters, which are essentially combination of standard deviations of some function of quantum phase, were introduced by Carruthers and Nieto [11] as a measure of quantum phase fluctuation. In recent past people have used Carruthers–Nieto parameters to study quantum phase fluctuations of coherent light interacting with a nonlinear nonabsorbing medium of inversion symmetry [10,12,16].

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But unfortunately any discussion regarding the physical meaning of these parameters are missing in the existing literature [10–12,16]. Present study aims to provide a physical meaning to these parameters. Here it is shown that the reduction of the parameter U with respect to its coherent state value corresponds to an antibunched state and it can be used as a measure of depth of nonclassicality. The importance of a systematic study of quantum phase fluctuation has increased with recent observations of quantum phase fluctuations in quantum computation [20,21] and superconductivity [15,22,23]. These observations along with the fact that the physical meaning of quantum phase fluctuation parameters are not clear have motivated us to study quantum phase fluctuation of pump mode photons in four wave mixing process, six wave mixing process and in second harmonic generation process. In next section we briefly introduce quantum phase fluctuations and attempt to provide a clear physical meaning to U parameter. We have presented a second order short time approximated operator solution of four wave mixing process in Section 3 and have used that to find out analytic expression for quantum phase fluctuation parameters. In Sections 4 and 5 we have given expressions for quantum phase fluctuation parameters for six wave mixing process and second harmonic generation respectively. Finally Section 6 is dedicated to conclusions.

2. Measures of quantum phase fluctuations: Understanding their physical meaning

Dirac [24] introduced the quantum phase operator with the assumption that the annihilation operator a can be factored out into a Hermitian function $f(N)$ of number operator N and a unitary operator U_1, which defines the Hermitian phase operator as $U_1 = \exp(i\phi)$. In this formalism explicit expression for a is given by

$$a = \exp(i\phi)N^{\frac{1}{2}} \tag{1}$$

which satisfies usual commutation relation $[a, a^\dagger] = 1$ only if the commutation relation

$$[N, \phi] = i \tag{2}$$

is satisfied. Again if (2) is true then the method of induction yield

$$\left[N, \phi^n\right] = in\phi^{n-1} = i\frac{d}{d\phi}\phi^n. \tag{3}$$

Therefore, for any polynomial function $P(\phi)$ of ϕ we have a commutation relation

$$\left[N, P(\phi)\right] = i\frac{dP(\phi)}{d\phi}. \tag{4}$$

Immediately after Dirac's introductory work it was realized that the uncertainty relation $\Delta N \Delta\phi \geqslant \frac{1}{2}$ associated with (2) has many problems [1]. For example we can note that it allows uncertainty in ϕ to (i.e. $\Delta\phi$) to be greater than 2π for $\Delta N < \frac{1}{4\pi}$. Later on Louisell [25] removed this problem by considering $P(\phi)$ present in (4) as a function of period 2π. Instead of bare phase operator he considered sine (S) and cosine (C) operators which satisfy

$$[N, C] = -iS \tag{5}$$

and

$$[N, S] = iC. \tag{6}$$

Therefore, the uncertainty relations associated with them are

$$\Delta N \Delta C \geqslant \frac{1}{2}\left|\langle S\rangle\right| \tag{7}$$

and

$$\Delta N \Delta S \geqslant \frac{1}{2}\left|\langle C\rangle\right|. \tag{8}$$

Susskind and Glogower [4] obtained the explicit form of S and C as

$$S = \frac{1}{2i}\left[\frac{1}{(N+1)^{\frac{1}{2}}}a - a^\dagger\frac{1}{(N+1)^{\frac{1}{2}}}\right] \tag{9}$$

and

$$C = \frac{1}{2}\left[\frac{1}{(N+1)^{\frac{1}{2}}}a + a^\dagger\frac{1}{(N+1)^{\frac{1}{2}}}\right]. \tag{10}$$

Now it is easy to see that the operators C and S satisfy

$$[C, S] = \frac{i}{2}P^0 \tag{11}$$

and

$$\langle C^2\rangle + \langle S^2\rangle = 1 - \frac{1}{2}\langle P^0\rangle \tag{12}$$

where $P^0 = |0\rangle\langle 0|$ is the projection onto the ground state. Squaring and adding (7) and (8) we obtain

$$(\Delta N)^2\left[(\Delta S)^2 + (\Delta C)^2\right]/\left[\langle S\rangle^2 + \langle C\rangle^2\right] \geqslant \frac{1}{4}. \tag{13}$$

Carruthers and Nieto [11] introduced (13) as measure of quantum phase fluctuation and named it as U parameter. They had also introduced two more parameters S and Q for the purpose of calculation of the phase fluctuations. To be precise Carruthers and Nieto defined following parameters as a measure of phase fluctuation:

$$U\left(\theta, t, |\alpha|^2\right) = (\Delta N)^2\left[(\Delta S)^2 + (\Delta C)^2\right]/\left[\langle S\rangle^2 + \langle C\rangle^2\right], \tag{14}$$

$$S\left(\theta, t, |\alpha|^2\right) = (\Delta N)^2(\Delta S)^2 \tag{15}$$

and

$$Q\left(\theta, t, |\alpha|^2\right) = S\left(\theta, t, |\alpha|^2\right)/\langle C\rangle^2 \tag{16}$$

where θ is the phase of the input coherent state $|\alpha\rangle$ (where $a|\alpha\rangle = \alpha|\alpha\rangle = |\alpha|\exp(i\theta)|\alpha\rangle$), t is the interaction time and $|\alpha|^2$ is the mean number of photon prior to the interaction. Later on these parameters draw more attention and many groups [10,12,16] have used these parameters as a measure of quantum phase fluctuation. Now one can, in principle, calculate the above parameters analytically by using an expression of time evolution of annihilation operator for a particular mode and it is already done in earlier works [10,12,16]. But the physical meanings of these parameters are not discussed till now. In order to obtain a physical meaning of these parameters we would like to associate phase fluctuation with total noise. The total noise of a quantum state is a measure of the total fluctuations of the amplitude. For a single mode quantum state having density matrix ρ it is defined as [17]

$$T(\rho) = (\Delta X)^2 + (\Delta \dot{X})^2. \tag{17}$$

In analogy to it we can define the total phase fluctuation as

$$T = (\Delta S)^2 + (\Delta C)^2. \tag{18}$$

Now using the relations (7), (8), (12), (13) and (18) we obtain:

$$\begin{aligned}(\Delta N)^2(\Delta S)^2 + (\Delta N)^2(\Delta C)^2 &\geqslant \frac{1}{4}\big(\langle S\rangle^2 + \langle C\rangle^2\big) \\ &= \frac{1}{4}\big(\langle S^2\rangle + \langle C^2\rangle - \big((\Delta S)^2 + (\Delta C)^2\big)\big)\end{aligned}$$

or

$$\frac{1}{4}\left(1 - \frac{1}{2}\langle P^0\rangle - \big((\Delta S)^2 + (\Delta C)^2\big)\right) \leqslant \big((\Delta S)^2 + (\Delta C)^2\big)(\Delta N)^2$$

or

$$U = \frac{((\Delta S)^2 + (\Delta C)^2)(\Delta N)^2}{(1 - \frac{1}{2}\langle P^0\rangle - ((\Delta S)^2 + (\Delta C)^2))} = \frac{T(\Delta N)^2}{(1 - \frac{1}{2}\langle P^0\rangle - T)} \geqslant \frac{1}{4}. \tag{19}$$

Since $[C, S] = \frac{i}{2}P^0$, therefore,

$$(\Delta C)^2(\Delta S)^2 \geqslant \frac{(\langle P^0\rangle)^2}{16}. \tag{20}$$

Now we can write

$$T = (\Delta C)^2 + (\Delta S)^2 \geqslant (\Delta C)^2 + \frac{\langle P^0\rangle^2}{16(\Delta C)^2}.$$

The function $T = (\Delta C)^2 + \frac{\langle P^0\rangle^2}{16(\Delta C)^2}$ has a clear minima at $(\Delta C)^2 = \frac{\langle P^0\rangle}{4}$, which corresponds to a coherent state and thus the total fluctuation in quantum phase variables $((\Delta C)^2 + (\Delta S)^2)$ cannot be reduced below its coherent state value $\frac{\langle P^0\rangle}{2}$. Now since $(\Delta N)^2$ is positive and the $U = \frac{T(\Delta N)^2}{(1-\frac{1}{2}\langle P^0\rangle - T)} = b(\Delta N)^2 \geqslant \frac{1}{4}$, therefore $b = \frac{T}{(1-\frac{1}{2}\langle P^0\rangle - T)}$ is positive. Again $a = (1 - \frac{\langle P^0\rangle}{2}) \geqslant \frac{1}{2}$ since the projection on to the ground state $\langle P^0\rangle = \langle |0\rangle\langle 0|\rangle \leqslant 1$. Thus the function b is of the form, $b = \frac{T}{a-T}$ where both a and b are positive. Under these conditions b increases monotonically with the increase in T. Thus the minima of T corresponds to the minima of b too and consequently, b is minimum for coherent state. In other words b cannot be reduced below its coherent state value. Therefore any reduction in $U = b(\Delta N)^2$ with respect to its coherent state value will mean a decrease in $(\Delta N)^2$ with respect to its coherent state counter part. Thus in SG formalism a decease in U will always mean an antibunching or sub-Poissonian photon statistics. But the converse is not true.

Let us see what happens in the other formalisms of the quantum phase problem. In case of Pegg–Barnett formalism [6], this notion of phase fluctuation is not valid since in this formalism $(\Delta C)^2 = (\Delta S)^2 = 0$ for $s \to \infty$, where s is dimension of the truncated Hilbert space in which the Pegg–Barnett sine and cosine operators are defined. But it can be shown that the BP formalism leads to same conclusion as in SG. To begin with we would like to note that the sine and cosine operators discussed so far have originated due to a rescaling of the photon annihilation and creation operators with the photon number operator. Another convenient way is to rescale an appropriate quadrature operator with the averaged photon number [3]. Barnett and Pegg did that and defined the exponential of phase operator E and its Hermitian conjugate $E^\dagger$ as [5]

$$\begin{aligned} E &= \left(\bar{N} + \frac{1}{2}\right)^{-1/2} a(t), \\ E^\dagger &= \left(\bar{N} + \frac{1}{2}\right)^{-1/2} a^\dagger(t) \end{aligned} \tag{21}$$

where $\bar{N}$ is the average number of photons present in the radiation field after interaction. The usual cosine and sine of the phase operator are defined in the following way

$$\begin{aligned} C &= \frac{1}{2}\big(E + E^\dagger\big), \\ S &= -\frac{i}{2}\big(E - E^\dagger\big). \end{aligned} \tag{22}$$

And this operators satisfy following relations

$$\langle C^2\rangle + \langle S^2\rangle = 1 \tag{23}$$

and

$$[C, S] = \frac{i}{2}\left(\bar{N} + \frac{1}{2}\right)^{-\frac{1}{2}}. \tag{24}$$

Therefore,

$$(\Delta C)^2(\Delta S)^2 \geqslant \frac{1}{16}\frac{1}{(\bar{N} + \frac{1}{2})}. \tag{25}$$

Now using Eqs. (23)–(25) and following the similar reasoning as we have used in Susskind–Glogower formalism it is straightforward to show the following:

(1) T has a minimum at $(\Delta C)^2 = \frac{1}{4}(\bar{N} + \frac{1}{2})^{-\frac{1}{2}}$, which corresponds to a coherent state and thus the total fluctuation in quantum phase variables $T = ((\Delta C)^2 + (\Delta S)^2)$ cannot be reduced below its coherent state value $\frac{1}{2}(\bar{N} + \frac{1}{2})^{-\frac{1}{2}}$.

(2) In BP formalism $U = \frac{T(\Delta N)^2}{(1-T)}$ and as both U and $(\Delta N)^2$ are positive so $b = \frac{T}{(1-T)}$ has to be positive and as a result b will monotonically increases with T. Consequently reduction in U compared with its value for a coherent state of the same photon number will imply sub-Poissonian photon statistics (antibunching) but not the vice versa.

From the above discussion it is clear that the physical meaning of U is same in both BP and SG formalism and a reduction in U with respect to its coherent state value implies antibunching. In next few sections we have verified our conclusions with specific examples. The examples are studied under BP formalism because of the inherent computational simplicity of these formalism over the others.

3. Time evolution of useful operators and phase fluctuation in four wave mixing process

The purpose of the present section is to calculate the phase fluctuations of pump mode photons in a four wave mixing process. We assume that initially, there is no photon in signal mode and stokes mode and a coherent light beam (laser) acts as pump which causes excitation followed by emissions. Thus our initial state is $|\alpha\rangle|0\rangle|0\rangle$. In the following subsection we have derived analytic expressions of useful operators connected to the four wave mixing process and have taken all the expectation values with respect to the initial state $|\alpha\rangle|0\rangle|0\rangle$.

3.1. Four wave mixing process

Four wave mixing may happen in different ways. One way is that two photon of frequency ω_1 are absorbed (as pump photon) and one photon of frequency ω_2 and another of frequency ω_3 are emitted. The Hamiltonian representing this particular four wave mixing process is

$$H = a^\dagger a\omega_1 + b^\dagger b\omega_2 + c^\dagger c\omega_3 + g\left(a^{\dagger 2}bc + a^2b^\dagger c^\dagger\right) \tag{26}$$

where a and $a^\dagger$ are creation and annihilation operators in pump mode which satisfy $[a, a^\dagger] = 1$, similarly b, $b^\dagger$ and c, $c^\dagger$ are creation and annihilation operators in stokes mode and signal mode respectively and g is the coupling constant. Substituting $A = ae^{i\omega_1 t}$, $B = be^{i\omega_2 t}$ and $C = ce^{i\omega_3 t}$ we can write the Hamiltonian (26) as

$$H = A^\dagger A\omega_1 + B^\dagger B\omega_2 + C^\dagger C\omega_3 + g\left(A^{\dagger 2}BC + A^2B^\dagger C^\dagger\right). \tag{27}$$

Since we know the Hamiltonian we can use Heisenberg's equation of motion

$$\dot{A} = \frac{\partial A}{\partial t} + i[H, A] \tag{28}$$

and short time approximation to find out the time evolution of the annihilation operator. From (27) and (28) we obtain

$$\dot{A} = iA\omega_1 - iA\omega_1 - i2gA^\dagger BC = -2igA^\dagger BC. \tag{29}$$

Similarly,

$$\dot{B} = -igA^2C^\dagger \tag{30}$$

and

$$\dot{C} = -igA^2B^\dagger. \tag{31}$$

We can find the second order differential of A using (28) as

$$\ddot{A} = \frac{\partial \dot{A}}{\partial t} + i[H, \dot{A}]$$

$$= 4g^2AB^\dagger BC^\dagger C - 2g^2A^\dagger A^2B^\dagger B - 2g^2A^\dagger A^2C^\dagger C - 2g^2A^\dagger A^2. \tag{32}$$

Substituting (29) and (32) in the Taylor's series expansion

$$f(t) = f(0) + t\left(\frac{\partial f(t)}{\partial t}\right)_{t=0} + \frac{t^2}{2!}\left(\frac{\partial^2 f(t)}{\partial t^2}\right)_{t=0} + \cdots \tag{33}$$

we obtain

$$A(t) = A - 2igtA^\dagger BC + \frac{g^2t^2}{2!}\left[4AB^\dagger BC^\dagger C - 2A^\dagger A^2B^\dagger B - 2A^\dagger A^2C^\dagger C - 2A^\dagger A^2\right]. \tag{34}$$

The Taylor series is valid when t is small, so this solution is valid for a short time and that is why it is called short time approximation. The above calculation is shown in detail as an example. Following the same procedure, we can find out time evolution of B and C or any other creation and annihilation operator that appears in the Hamiltonian of matter field interaction. This is a very strong technique since this straight-forward prescription is valid for any optical process where interaction time is short.

Now, from Eq. (34) we can easily derive expression for $N(t)$ as

$$N(t) = A^\dagger A + 2igt\left(A^2B^\dagger C^\dagger - A^{\dagger 2}BC\right) + 4g^2t^2\left(2A^\dagger AB^\dagger BC^\dagger C + B^\dagger BC^\dagger C\right) - 2g^2t^2\left(A^{\dagger 2}A^2B^\dagger B + A^{\dagger 2}A^2C^\dagger C + A^{\dagger 2}A^2\right). \tag{35}$$

Eq. (35) can be used to obtain

$$\bar{N} = \langle N\rangle = |\alpha|^2 - 2g^2t^2|\alpha|^4. \tag{36}$$

Now taking the square of $\langle N\rangle$, one can easily find

$$\langle N\rangle^2 = |\alpha|^4 - 4g^2t^2|\alpha|^6. \tag{37}$$

On the other hand, using (35) and operator ordering techniques we can show that

$$\left\langle N^2(t)\right\rangle = |\alpha|^2 - |\alpha|^4 - g^2t^2\left[4|\alpha|^6 + 8|\alpha|^4\right]. \tag{38}$$

Using (37) and (38) we can write

$$(\Delta N)^2 = |\alpha|^2 - 8g^2t^2|\alpha|^4. \tag{39}$$

As we have already discussed in the introduction, the reduction of photon number fluctuation below its coherent state value (Poisson level) corresponds to an antibunched state. So the condition for the existence of antibunching can be expressed as

$$d = (\Delta N)^2 - \bar{N} < 0. \tag{40}$$

From (36) and (39) it is clear that

$$d = -6g^2t^2|\alpha|^4 \tag{41}$$

is always negative and we always obtain an antibunched state. Now let us check whether this antibunching phenomenon really causes reduction of U or not. In order to do so we substitute (34) in (22) and obtain

$$C = \frac{1}{2}\left(\bar{N}+\frac{1}{2}\right)^{-\frac{1}{2}}\big[A + A^\dagger - 2igtA^\dagger BC + 2igtAB^\dagger C^\dagger + g^2t^2\big\{2AB^\dagger BC^\dagger C + 2A^\dagger B^\dagger BC^\dagger C - A^\dagger A^2C^\dagger C - A^{\dagger 2}AC^\dagger C - A^\dagger A^2B^\dagger B - A^{\dagger 2}AB^\dagger B - A^\dagger A^2 - A^{\dagger 2}A\big\}\big] \quad (42)$$

and

$$S = -\frac{i}{2}\left(\bar{N}+\frac{1}{2}\right)^{-\frac{1}{2}}\big[A - A^\dagger - 2igtA^\dagger BC - 2igtAB^\dagger C^\dagger + g^2t^2\big\{2AB^\dagger BC^\dagger C - 2A^\dagger B^\dagger BC^\dagger C - A^\dagger A^2C^\dagger C + A^{\dagger 2}AC^\dagger C - A^\dagger A^2B^\dagger B + A^{\dagger 2}AB^\dagger B - A^\dagger A^2 + A^{\dagger 2}A\big\}\big]. \quad (43)$$

Using (42) and (43) the expectation values of the operators C and S can be obtained as

$$\langle C\rangle = \frac{1}{2}\left[\left(\bar{N}+\frac{1}{2}\right)^{-\frac{1}{2}}\big\{\big(\alpha - \alpha|\alpha|^2g^2t^2\big) + \big(\alpha^* - \alpha^*|\alpha|^2g^2t^2\big)\big\}\right] \quad (44)$$

and

$$\langle S\rangle = -\frac{i}{2}\left[\left(\bar{N}+\frac{1}{2}\right)^{-\frac{1}{2}}\big\{\big(\alpha - \alpha|\alpha|^2g^2t^2\big) - \big(\alpha^* - \alpha^*|\alpha|^2g^2t^2\big)\big\}\right]. \quad (45)$$

Again, the square of the averages are

$$\langle C\rangle^2 = \frac{1}{4}\left(\bar{N}+\frac{1}{2}\right)^{-1}\big[\alpha^2 + \alpha^{*2} + 2|\alpha|^2 - g^2t^2\big\{2\alpha^2|\alpha|^2 + 2\alpha^{*2}|\alpha|^2 + 4|\alpha|^4\big\}\big] \quad (46)$$

and

$$\langle S\rangle^2 = -\frac{1}{4}\left(\bar{N}+\frac{1}{2}\right)^{-1}\big[\alpha^2 + \alpha^{*2} - 2|\alpha|^2 - g^2t^2\big\{2\alpha^2|\alpha|^2 + 2\alpha^{*2}|\alpha|^2 - 4|\alpha|^4\big\}\big]. \quad (47)$$

Squaring C and S and taking expectation value with respect to initial state, we have

$$\langle C^2\rangle = \frac{1}{4}\left(\bar{N}+\frac{1}{2}\right)^{-1}\big[\alpha^2 + \alpha^{*2} + 2|\alpha|^2 + 1 - g^2t^2\big\{\alpha^2 + \alpha^{*2} + 2\alpha^{*2}|\alpha|^2 + 2\alpha^2|\alpha|^2 + 4|\alpha|^4 + 4|\alpha|^2\big\}\big], \quad (48)$$

$$\langle S^2\rangle = -\frac{1}{4}\left(\bar{N}+\frac{1}{2}\right)^{-1}\big[\alpha^2 + \alpha^{*2} - 2|\alpha|^2 - 1 - g^2t^2\big\{\alpha^2 + \alpha^{*2} + 2\alpha^{*2}|\alpha|^2 + 2\alpha^2|\alpha|^2 - 4|\alpha|^4 - 4|\alpha|^2\big\}\big]. \quad (49)$$

Using Eqs. (46)–(49) the second order variances ($(\Delta C)^2$ and $(\Delta S)^2$) of C and S can be calculated as

$$(\Delta C)^2 = \frac{1}{4}\left(\bar{N}+\frac{1}{2}\right)^{-1}\big[1 - g^2t^2\big\{\alpha^2 + \alpha^{*2} + 4|\alpha|^2\big\}\big] \quad (50)$$

and

$$(\Delta S)^2 = -\frac{1}{4}\left(\bar{N}+\frac{1}{2}\right)^{-1}\big[-1 - g^2t^2\big\{\alpha^2 + \alpha^{*2} - 4|\alpha|^2\big\}\big]. \quad (51)$$

Interestingly, $\bar{N}$ depends on the coupling constant g and on the free evolution time t. Now Eqs. (14)–(16) assume the following forms,

$$U\big(\theta, t, |\alpha|^2\big) = \frac{1}{2}\left\{\frac{1 - 12g^2t^2|\alpha|^2}{1 - 2g^2t^2|\alpha|^2}\right\}, \quad (52)$$

$$S\big(\theta, t, |\alpha|^2\big) = \frac{1}{4}\left(\bar{N}+\frac{1}{2}\right)^{-1} \times \big[|\alpha|^2 + g^2t^2\big\{\alpha^2|\alpha|^2 + \alpha^{*2}|\alpha|^2 - 12|\alpha|^4\big\}\big] = \frac{1}{4}\left(|\alpha|^2 - 2g^2t^2|\alpha|^4 + \frac{1}{2}\right)^{-1} \times \big[|\alpha|^2 + 2|\alpha|^4g^2t^2\{\cos 2\theta - 6\}\big] \quad (53)$$

and

$$Q\big(\theta, t, |\alpha|^2\big) = \frac{|\alpha|^2 + g^2t^2\{\alpha^2|\alpha|^2 + \alpha^{*2}|\alpha|^2 - 12|\alpha|^4\}}{\alpha^2 + \alpha^{*2} + 2|\alpha|^2 - 2g^2t^2\{\alpha^2|\alpha|^2 + \alpha^{*2}|\alpha|^2 + 2|\alpha|^4\}} = \frac{1 + 2|\alpha|^2g^2t^2\{\cos 2\theta - 6\}}{2(\cos 2\theta + 1)(1 - 2|\alpha|^2g^2t^2)}. \quad (54)$$

Hence Eqs. (52)–(54) are our desired results. In the derivation of Eq. (54), we have assumed $|\alpha|^2 \neq 0$. Now, $U_0 = \frac{1}{2}$, $S_0 = \frac{1}{4}|\alpha|^2(|\alpha|^2 + \frac{1}{2})^{-1}$ and $Q_0 = \frac{1}{2\cos 2\theta}$ are the initial (i.e. $\lambda = 0$) values of U, S and Q respectively. Thus U_0, S_0 and Q_0 signify the information about the phase of the input coherent light. As we have already discussed, any reduction of U will correspond to antibunching of photon. Now the negativity of (41) manifests the existence of antibunching. It is also clear from (41) that the depth of antibunching (up to second order in coupling constant) decreases monotonically with the increase in initial photon number $|\alpha|^2$. These facts are manifested in (52) which decreases monotonically with respect to its coherent state value. The suitable choice of t and θ may also cause the enhancement and reduction of S and Q parameters compared to their initial values. It is to be noted that the parameters U, S and Q contain the secular terms proportional to t. However, it is not a serious problem since the product g^2t^2 is small. Eqs. (52)–(54) are good enough to have the flavor of analytical results.

4. Six wave mixing process

Similar to four wave mixing process six wave mixing process may also happen in different ways. One way is one in which two photon of frequency ω_1 are absorbed (as pump photon) and three photon of frequency ω_2 and another of frequency ω_3 are emitted. The Hamiltonian representing this particular six wave mixing process is

$$H = A^\dagger A\omega_1 + B^\dagger B\omega_2 + C^\dagger C\omega_3 + g\big(A^{\dagger 2}B^3C + A^2B^{\dagger 3}C^\dagger\big). \tag{55}$$

By following the technique elaborated in the last section we can obtain the solution of (55) as

$$\begin{aligned}A(t) &= A - 2igtA^\dagger B^3C \\ &\quad + g^2t^2\big[2AB^{\dagger 3}B^3C^\dagger C - 9A^\dagger A^2B^{\dagger 2}B^2C^\dagger C \\ &\quad - 18A^\dagger A^2B^\dagger BC^\dagger C - A^\dagger A^2B^{\dagger 3}B^3 - 9A^\dagger A^2B^{\dagger 2}B^2 \\ &\quad - 18A^\dagger A^2B^\dagger B - 6A^\dagger A^2C^\dagger C - 6A^\dagger A^2\big].\end{aligned} \tag{56}$$

Now from Eq. (56) we obtain

$$\bar{N} = \langle N\rangle = |\alpha|^2 - 12g^2t^2|\alpha|^4 \tag{57}$$

and

$$d = -12g^2t^2|\alpha|^4 \tag{58}$$

is always negative. This fact indicates the presence of antibunching. By using (55) and (56) we can write the Carruthers–Nieto quantum phase fluctuation parameters as

$$U\big(\theta, t, |\alpha|^2\big) = \frac{1}{2}\left\{\frac{1 - 72g^2t^2|\alpha|^2}{1 - 12g^2t^2|\alpha|^2}\right\}, \tag{59}$$

$$\begin{aligned}S\big(\theta, t, |\alpha|^2\big) &= \frac{1}{4}\left(\bar{N} + \frac{1}{2}\right)^{-1} \\ &\quad \times \big[|\alpha|^2 + 6g^2t^2\{\alpha^2|\alpha|^2 + \alpha^{*2}|\alpha|^2 - 12|\alpha|^4\}\big] \\ &= \frac{1}{4}\left(|\alpha|^2 - 12g^2t^2|\alpha|^4 + \frac{1}{2}\right)^{-1} \\ &\quad \times \big[|\alpha|^2 + 12|\alpha|^4g^2t^2\{\cos 2\theta - 6\}\big]\end{aligned} \tag{60}$$

and

$$\begin{aligned}Q\big(\theta, t, |\alpha|^2\big) &= \frac{|\alpha|^2 + 6g^2t^2\{\alpha^2|\alpha|^2 + \alpha^{*2}|\alpha|^2 - 12|\alpha|^4\}}{\alpha^2 + \alpha^{*2} + 2|\alpha|^2 - 12g^2t^2\{\alpha^2|\alpha|^2 + \alpha^{*2}|\alpha|^2 + 2|\alpha|^4\}} \\ &= \frac{1 + 12|\alpha|^2g^2t^2\{\cos 2\theta - 6\}}{2(\cos 2\theta + 1)(1 - 12|\alpha|^2g^2t^2)}.\end{aligned} \tag{61}$$

From (59) it is clear that with the increase initial photon number, U reduces monotonically from its coherent state value indicating the existence of antibunching. As it is appears from (58) and (59), if the photon number distribution is more nonclassical (i.e. degree of antibunching is more) then U is less.

5. Second harmonic generation

Second harmonic generation is a process in which two photons of frequency ω are absorbed and a photon of frequency 2ω is emitted. The Hamiltonian describing this process is

$$H = \hbar\omega N_1 + 2\hbar\omega N_2 + hg\big(a_2^\dagger a_1^2 + a_1^{\dagger 2}a_2\big). \tag{62}$$

Now following the same procedure as we have done in Section 3 we can obtain

$$A(t) = a_1 - 2igta_1^\dagger a_2 + 2g^2t^2\left(a_2^\dagger a_2a_1 - \frac{1}{2}a_1^\dagger a_1^2\right) \tag{63}$$

and

$$\bar{N} = \langle N\rangle = |\alpha|^2 - 2g^2t^2|\alpha|^4. \tag{64}$$

This optical process shows antibunching since from (63) and (40) we obtain

$$d = -2g^2t^2|\alpha|^4. \tag{65}$$

The fact that the existence of antibunching appears through the reduction of quantum phase fluctuation parameter U will be clear from the following expression

$$U\big(\theta, t, |\alpha|^2\big) = \frac{1}{2}\left\{\frac{1 - 4g^2t^2|\alpha|^2}{1 - 2g^2t^2|\alpha|^2}\right\}. \tag{66}$$

The other phase fluctuation parameters are

$$\begin{aligned}S\big(\theta, t, |\alpha|^2\big) &= \frac{1}{4}\left(\bar{N} + \frac{1}{2}\right)^{-1} \\ &\quad \times \big[|\alpha|^2 + g^2t^2\{\alpha^2|\alpha|^2 + \alpha^{*2}|\alpha|^2 - 4|\alpha|^4\}\big] \\ &= \frac{1}{4}\left(|\alpha|^2 - 2g^2t^2|\alpha|^4 + \frac{1}{2}\right)^{-1} \\ &\quad \times \big[|\alpha|^2 + 2|\alpha|^4g^2t^2\{\cos 2\theta - 2\}\big]\end{aligned} \tag{67}$$

and

$$\begin{aligned}Q\big(\theta, t, |\alpha|^2\big) &= \frac{|\alpha|^2 + g^2t^2\{\alpha^2|\alpha|^2 + \alpha^{*2}|\alpha|^2 - 4|\alpha|^4\}}{\alpha^2 + \alpha^{*2} + 2|\alpha|^2 - 2g^2t^2\{\alpha^2|\alpha|^2 + \alpha^{*2}|\alpha|^2 + 2|\alpha|^4\}} \\ &= \frac{1 + 2|\alpha|^2g^2t^2\{\cos 2\theta - 2\}}{2(\cos 2\theta + 1)(1 - 2|\alpha|^2g^2t^2)}.\end{aligned} \tag{68}$$

6. Conclusion

The physical meaning of the Carruthers–Nieto symmetric quantum phase fluctuation parameter (U) has been obtained in Susskind–Glogower and Barnett–Pegg formalism of quantum phase and it is shown that the reduction of phase fluctuation parameter U with respect to its coherent state value corresponds to an antibunched (sub-Poissonian) state. The idea is also verified by analytical study of quantum phase fluctuations in three different optical processes, such as four wave mixing, six wave mixing and second harmonic generation. The study shows that the reduction of phase fluctuations compared to their initial values are possible in all these cases. It is also shown that the symmetric product in uncertainty U is independent of θ in all three cases. This is in sharp contrast to earlier work of Pathak and Mandal [16]. In general S and Q can be tuned by turning the input phase θ but interestingly Q can be tuned even for a vacuum field (when $|\alpha|^2 = 0$) while $S = 0$ for such a situation. U is found to reduce monotonically with initial photon number $|\alpha|^2$. The rate of reduction is maximum in six wave mixing process. The amount of reduction in U is directly related to the nonclassical depth of antibunching and U can be considered as an indirect measure of amount of antibunching or the depth of nonclassicality. A huge reduction of U is possible with the increase of the initial photon number $|\alpha|^2$. However, care should be taken about the condition of the solution during such increase.

In the earlier works [10,12], U was enhanced compared to its initial values as $|\alpha|^2$ increases. In the works of Pathak and Mandal [16], it was reported that U may be reduced from its coherent state value but neither the physical meaning of this reduction nor its relation with the antibunching was explored. Hence, the present results are in sharp contrast with the earlier studies [10,12], and also provide a clear meaning to earlier work of Pathak and Mandal [16].

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References

[1] R. Lynch, Phys. Rep. 256 (1995) 367.
[2] Z. Ficek, M.R. Wahiddin, Quantum Optics Fundamental and Applications, IIUM, Kuala Lumppur, 2004 (Chapter 5).
[3] V. Perinova, A. Luks, J. Perina, Phase in Optics, World Scientific, Singapore, 1998 (Chapter 4).
[4] L. Susskind, J. Glogower, Physics 1 (1964) 49.
[5] S.M. Barnett, D.T. Pegg, J. Phys. A 19 (1986) 3849.
[6] D.T. Pegg, S.M. Barnett, Phys. Rev. A 41 (1989) 3427.
[7] H.-Y. Fan, H.R. Zaidi, Opt. Commun. 68 (1988) 143.
[8] B.C. Sanders, S.M. Barnett, P.L. Knight, Opt. Commun. 58 (1986) 290.
[9] D. Yao, Phys. Lett. A 122 (1987) 77.
[10] C.C. Gerry, Opt. Commun. 63 (1987) 278.
[11] P. Carruthers, M.M. Nieto, Rev. Mod. Phys. 40 (1968) 411.
[12] R. Lynch, Opt. Commun. 67 (1988) 67.
[13] R. Lynch, J. Opt. Soc. Am. B 4 (1987) 1723.
[14] J.A. Vaccaro, D.T. Pegg, Opt. Commun. 105 (1994) 335.
[15] Y.K. Tsui, Phys. Rev. A 47 (1993) 12296.
[16] A. Pathak, S. Mandal, Phys. Lett. A 272 (2000) 346.
[17] A. Orlowski, Phys. Rev. A 48 (1993) 727.
[18] V.V. Dodonov, J. Opt. B Quant. Semiclass. Opt. 4 (2002) R1.
[19] R. Hanbury-Brown, R.Q. Twiss, Nature 177 (1956) 27.
[20] A.B. Klimov, et al., J. Phys. A 37 (2004) 4097.
[21] L.L. Sanchez-Soto, et al., Phys. Rev. A 66 (2002) 042112.
[22] M.M. Nieto, Phys. Rev. 167 (1968) 416.
[23] I. Iguchi, T. Yamaguchi, A. Sugimato, Nature 412 (2001) 420.
[24] P.A.M. Dirac, Proc. R. Soc. London, Ser. A 114 (1927) 243.
[25] W.H. Louisell, Phys. Lett. 7 (1963) 60.